UNIVERSITIES OF MANCHESTER LIVERPOOL
LEEDS SHEFFIELD AND BIRMINGHAM

Joint Matriculation Board

General Certificate of Education

# MATHEMATICS

## SCHOLARSHIP

FRIDAY 22 JUNE 1956, 9.30-12.30

*Answer not more than* **ten** *questions. Full marks may be obtained for complete answers to* **seven** *questions.*

*Candidates need not confine their attention to the questions which correspond to the Alternative they offer in Mathematics, Advanced, Paper II. For the information of candidates these questions have A, B or C respectively in front of the number.*

**1.** (*a*) The sum of the first $n$ terms of a series is $2n+3n^2$. Show that the series is an arithmetic progression, and find the sum of the first $n$ terms occupying odd places in the series.

(*b*) Sum to infinity the series

$$1+(1+r)s+(1+r+r^2)s^2+\ldots,$$

and state the conditions under which the sum exists.

**2.** (*a*) Find the real values of $x$ satisfying the equation

$$(x^2+x-\tfrac{1}{3})(x^2+x+\tfrac{1}{3}) = \tfrac{1}{3}.$$

(*b*) Solve the simultaneous equations

$$\sin 2x - \cos y = 0,$$
$$\cos x - \sin 2y = 0,$$

subject to the conditions $0 < x < \pi/2$, $0 < y < \pi/2$.

**3.** Two vertical walls of a building meet at an angle $\alpha$. The tops of the walls are horizontal, and they support two plane faces of roof which are each inclined at an angle $\beta$ to the horizontal. Show that the inclination to the horizontal of the line in which the roof faces meet is $\tan^{-1}(\sin \frac{1}{2}\alpha \tan \beta)$, and that the angle between the faces is $2 \cos^{-1}(\cos \frac{1}{2}\alpha \sin \beta)$.

**4.** A straight line of gradient $m$ is drawn through the fixed point $x = a$, $y = b$ to meet the coordinate axes at $P$ and $Q$. Show that, as $m$ varies, the length $PQ$ has a stationary value when $m = -(b/a)^{\frac{1}{3}}$, and find this value.

**5.** Find the equation of the normal at the point $t$ on the parabola $x = at^2$, $y = 2at$, and determine the coordinates of the point in which the normal again intersects the parabola. Find also the equation of the locus of the point of intersection of the tangents at these two points as $t$ varies.

**A6.** An open circular cylindrical can, of radius $2a$ and height $3a$, stands with its base on a horizontal table and contains, projecting from it, a uniform rod $ABC$ of length $ma$. The rod is in contact at $A$ with both the bottom and the curved surface of the can and leans against the rim at $B$ so that $AB$ passes through the axis of the can. All contacts are smooth. The weight of the can is $nW$ and its centre of gravity is on its axis, and the weight of the rod is $W$. Show that, whatever the value of $n$, the rod will slip if $m$ is greater than 125/8.

Determine, in terms of $m$, the minimum value of $n$ necessary to prevent the can from tipping.

**A7.** A bucket has the shape of a frustum of a cone, its base being of radius $a$, the rim of radius $2a$, and the height $2a$. Show that the cross-sectional area of the bucket at a height $y$ above its base is $\pi(a+\frac{1}{2}y)^2$.

The bucket is initially full of water, and there is a small hole in its base which allows water to leak out in a jet whose cross-sectional area is $\pi a^2/150$. The velocity of the water in the jet is $(2gy)^{\frac{1}{2}}$ when the height of the remaining water is $y$. Determine how long it takes for the bucket to empty.

**A8.** The engine of a train of mass $M$ tons can exert a maximum pull of $pM$ tons weight, and the greatest horse-power that it can develop is $\frac{224}{55}\, hMV$, where $V$ is a velocity; $h$ and $V$ are constants. The train experiences a constant resistance $rM$ tons weight. In a certain test the train is started from rest and the engine is brought up to its full horse-power as quickly as possible. Show that this takes a time $t$ minutes, where

$$t = \frac{1}{60g}\,\frac{hV}{p(p-r)}\,.$$

If the engine continues to develop its full horse-power for a further $t$ minutes, find the total work done by the engine in foot-pounds-weight.

**A9.** A light elastic spring $AB$ is constrained to lie in a vertical line and is fixed at its lower end $B$. It carries a small cup of mass $m$ at $A$ in which there is a particle of mass $m$, and this load compresses the spring by the small amount $a$. In a motion of small amplitude about the position of equilibrium the compression of the spring at time $t$ is $a+x$. Establish the equations

$$\frac{d^2x}{dt^2} = -\frac{g}{a}\,x,$$

$$\left(\frac{dx}{dt}\right)^2 + \frac{g}{a}\,x^2 = \text{const.}$$

If the system is released from rest when $x = 3a$, show that the particle will leave the cup when $x = -a$, and determine the height to which it will rise above the latter position.

**A10.** A small elastic sphere is projected from a point $A$ on a smooth horizontal floor. It strikes normally a vertical wall distant $a$ from $A$, rebounds, and after one bounce on the floor returns to $A$. If $e$ is the coefficient of restitution at each impact, prove that $e = \frac{1}{2}$, and find at what distance from the wall the sphere will finally come to rest if it continues to bounce on the floor.

**B11.** The output, $N$ articles per day, of a machine slows down in such a way that the rate of decrease of $N$ is proportional to the product of $N$ and the total time $t$ that the machine has been in use. Express this statement as a differential equation and solve it for $N$ in terms of $t$ and any necessary constants.

Initially the output was 1,000 articles per day but after 50 days it has dropped to 950 articles per day. Calculate how much longer the machine will be kept in use if it is to be discarded as soon as its output falls to 500 articles per day.

**B12.** In a cricket match each over consists of six balls and the bowling is opened by two players $A$ and $B$, the player $A$ bowling the first over. If the probability that $A$ takes a wicket with each ball is $\frac{1}{12}$ and the corresponding probability for $B$ is $\frac{1}{15}$, show that the probability that $A$ takes at least one wicket in his first over is approximately $\frac{2}{5}$ whilst the probability that he takes a wicket before $B$ is approximately $\frac{2}{3}$.

**B13.** The marks obtained by the $n$ candidates who passed an examination but did not reach the credit standard ranged from 68 to 76. They were converted to a range of 50 to 60 by reading off the new mark, $y$, corresponding to an old mark, $x$, from the straight line graph joining the point (68, 50) to the point (76, 60). Find a formula for $y$ in terms of $x$ and deduce relationships between

(i) the new mean, $\bar{y}$, and the old mean, $\bar{x}$,

(ii) the new standard deviation, $s'$, and the old standard deviation, $s$.

**B14.** Calculate the mean and variance of the binomial distribution in which the distribution of the relative frequencies (the total frequency is unity) of 0, 1, 2, ...... $n$ successes in $n$ events consists of the terms in the expansion of $(q+p)^n$, where $q+p = 1$, $q$ being the chance of failure and $p$ of success in each event.

Two ounces of seeds of yellow wall-flowers are thoroughly mixed before sowing with eight ounces of seeds of red wall-flowers and eventually the plants are bedded out in rows of twenty. Estimate the mean and variance of the number of yellow flowers in each row.

**B15.** Calculate the mean value of $10 \cos \theta$ over the range $0 \leqslant \theta \leqslant \frac{\pi}{2}$, giving the result in terms of $\pi$.

The distance between adjacent printed lines on a sheet of writing paper is half an inch. The paper is laid flat on a horizontal table and a straight needle 5 inches long is thrown at random 40 times on to the paper. The following table shows the frequency distribution of the number of intersections of the printed lines made by the needle as it lies on the table after each throw:

| Number of Intersections | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Frequency | 3 | 4 | 1 | 3 | 1 | 1 | 4 | 4 | 6 | 6 | 7 |

Calculate the mean number of intersections.

Combine the mean value obtained in the first part of this question with the mean obtained in the second to estimate the value of $\pi$, explaining by means of a diagram the principle involved.

**C16.** Prove that the equation

$$\frac{x^2-x}{x^2+x-1} = p,$$

where $p \neq 1$, always has two real roots. If the roots are $\alpha$ and $\beta$, show that $\alpha(\alpha-1)$ and $\beta(\beta-1)$ have opposite signs unless $p = 0$.

Determine the range of values of $p$ for which $\alpha$ and $\beta$ are both positive.

**C17.** (*a*) Prove that if $f(\alpha) = 0$, then $(x-\alpha)$ is a factor of the polynomial $f(x)$.

(*b*) The polynomial $f(x)$ is divided by $x^2-x$, and the remainder is $A+Bx$. Determine the constants $A$ and $B$.

(*c*) If $g(x)$ is a polynomial of degree $n$ which vanishes for $x = \alpha_1, \alpha_2, \ldots, \alpha_n$, show that

$$\frac{g'(x)}{g(x)} = \sum_{r=1}^{n} \frac{1}{x-\alpha_r},$$

where $g'(x)$ denotes the derivative of $g(x)$.

**C18.** Prove the following statements, $k$ and $n$ being integers:

(i) the fourth power of an odd integer is of the form $16k+1$,

(ii) $n^{12} = 13k$ or $13k+1$,

(iii) $n^7-n = 42k$.

**C19.** Each of the lines $OA$, $OB$, $OC$ is perpendicular to the other two, and their lengths are $a$, $b$, $c$ respectively. Find the radius of the sphere that can be drawn through $O$, $A$, $B$ and $C$.

Spheres pass through $C$ and the mid-point of $BC$ to touch the plane $AOB$. Show that the locus of their points of contact with the plane is a circle, and find its radius.

In the rectangular box of which $OA$, $OB$, $OC$ are concurrent edges, show that the shortest distance between the diagonal through $O$ and an edge of length $c$ which does not meet this diagonal is $ab(a^2+b^2)^{-\frac{1}{2}}$.

**C20.** The perpendiculars from the foci $F, F'$ of the ellipse $x^2/a^2+y^2/b^2=1$ meet the tangent at the point $(a\cos\theta,\ b\sin\theta)$ in $Y, Y'$. Show that the $y$-coordinate of the centre of the circle which has $YY'$ as a diameter is

$$\frac{a^2 b\sin\theta}{a^2\sin^2\theta+b^2\cos^2\theta},$$

and find the length of the radius.

Prove that if the circle touches $FF'$,

$$\sin^2\theta=\frac{b^4}{(a^2-b^2)^2}.$$

**C21.** (*a*) If $I_n=\int_0^{\pi/4}\tan^n\theta\,d\theta$, find the relation between $I_n$ and $I_{n-2}$, where $n\geqslant 2$.

(*b*) Show that

$$\int_0^{\pi/2} f(\sin\theta)\,d\theta=\int_0^{\pi/2} f(\cos\theta)\,d\theta,$$

and evaluate

$$\int_0^{\pi/2}\frac{\sin\theta}{\sin\theta+\cos\theta}\,d\theta.$$

**C22.** (*a*) The curve $y = f(x)$ passes through the point (3, 1), and its gradient at the point $(x, y)$ is given by the differential equation

$$\frac{dy}{dx} = \frac{1}{2}\left(1 - \frac{y}{x}\right).$$

Find $f(x)$.

(*b*) By substituting $y = z^{-\frac{1}{2}}$, or otherwise, solve the differential equation

$$\frac{dy}{dx} = y - xe^{-2x}y^3.$$