UNIVERSITIES OF MANCHESTER LIVERPOOL
LEEDS SHEFFIELD AND BIRMINGHAM

Joint Matriculation Board

General Certificate of Education

MATHEMATICS. PAPER I

ADVANCED

MONDAY 24 JUNE 1957, 9.30-12.30

Answer **all** *questions in* SECTION (1) *and* **four** *questions from* SECTION (2) *in the same answer-book.*

I sheet of graph paper supplied. Additional sheets will be supplied on request but **all** *sheets issued must be placed within the answer-book and handed in to the Supervisor.*

Logarithm tables are also provided.

SECTION (1)

Answer **all** *questions in this section.*

✓ **1.** Express the function

$$\frac{1+2x+3x^2}{(1-x)(1+x^2)}$$

in partial fractions.

If x is so small that powers higher than the third may be neglected, expand the function in the form

$$A+Bx+Cx^2+Dx^3.$$

✓ **2.** Given that $3\cos\theta-\sin\theta=R\cos(\theta+\alpha)$ obtain the values of R and α, where R is a positive constant and α an acute angle.

Hence, or otherwise, find the values of θ lying between 0° and 360° which satisfy the equation

$$3\cos\theta-\sin\theta=2.$$

✓ **3.** The coordinates of the points A and B are $(-2, 2)$ and $(3, 1)$ respectively. Show that the equation of the circle which has AB as a diameter is

$$x^2+y^2-x-3y-4=0.$$

If A and B are opposite corners of a square, find the coordinates of the other two corners.

✓ **4.** A sports field is to have the shape of a rectangular area $ABCD$ with semi-circular areas at opposite ends on BC and AD as diameters. Its perimeter is to be 440 yards long and the area of the rectangle $ABCD$ is to be a maximum. Find the dimensions of the rectangle. (Take π as 22/7.)

✓ **5.** If $y = e^{-x}\cos x$, determine the three values of x between 0 and 3π for which $dy/dx = 0$. Show that the corresponding values of y form a geometric progression with common ratio $-e^{-\pi}$.

✓ **6.** (*a*) Prove that $\int_0^{\pi/2} (\sin x+\cos x)^2 dx = \frac{\pi}{2}+1.$

(*b*) Integrate xe^{2x} with respect to x.

Section (2)

Answer **four** *questions from this section.*

7. Given that $c(x-a)^p+b = 2x^3-12x^2+24x-13$, obtain the values of c, a, p and b.

Hence, or otherwise, find the real root of the equation

$$2x^3-12x^2+24x-13=0,$$

giving your answer to three decimal places.

Sketch the graph of $y = 2x^3-12x^2+24x-13$.

8. A chord AB of a circle subtends an angle θ radians at the centre C of the circle. If the chord divides the area of the circle in the ratio 1:3, show that

$$\sin\theta = \theta - \frac{\pi}{2}.$$

By drawing the graphs of $y = \sin\theta$ and $y = \theta - \dfrac{\pi}{2}$ between 0 and π on the same diagram, find the angle ACB in degrees.

✓ **9.** The line $y = mx + c$ is a tangent to the parabola $y^2 = 4ax$. Prove that $c = a/m$ and find the coordinates of the point of contact in terms of a and m.

The tangents at the points P and Q of the parabola $y^2 = 4ax$ intersect at right angles at T and the normals at P and Q intersect at N. Prove that TN is parallel to the axis of the parabola.

✓ **10.** From a point A on a mountain the angle of elevation of the summit C is α. From a point B higher up the mountain and in the vertical plane through AC the angle of elevation of C is β. The straight line AB is c feet long and the angle of elevation of B from A is θ. If the difference in altitude between C and B is h feet, show that

$$h = c\sin\beta\sin(\alpha-\theta)\operatorname{cosec}(\beta-\alpha).$$

By regarding c, α and θ as constants and differentiating h with respect to β, show that the percentage error in the calculated value of h caused by a small error $\delta\beta$ in β is approximately

$$100\{\cot\beta - \cot(\beta-\alpha)\}\delta\beta.$$

11. Show that the area enclosed between the curve $y = \tan x$ and the lines $y = 0$ and $x = \frac{1}{4}\pi$ is $\frac{1}{2}\log_e 2$.

If the above area is rotated through four right angles about the line $y = 0$, prove that the volume generated is $\pi(4-\pi)/4$.

12. Assuming that $|x|<1$, write down

(i) the sum of the infinite geometric series

$$1+x^2+x^4+\ldots,$$

(ii) the first three terms of the series for $\log_e(1+x)$.

Obtain the first two terms of the series for

$$\tfrac{1}{2}\log_e\left(\frac{1+x}{1-x}\right).$$

Assuming also that x is positive, show that the sum of the remaining terms of this series is less than

$$\frac{x^5}{5(1-x^2)}.$$